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RESEARCH ARTICLE

Approximate Controllability of Fractional Nonlocal Delay Semilinear Systems in Hilbert Spaces

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We study the existence and approximate controllability of a class of fractional nonlocal delay semilinear differential systems in a Hilbert space. The results are obtained by using semigroup theory, fractional calculus, and Schauder's fixed point theorem. Multi-delay controls and a fractional nonlocal condition are introduced. Furthermore, we present an appropriate set of sufficient conditions for the considered fractional nonlocal multi-delay control system to be approximately controllable. An example to illustrate the abstract results is given.

Keywords: approximate controllability; fractional multi-delay control system; fractional nonlocal condition; Schauder's fixed point theorem; semigroups.

1 Introduction

We are concerned with the fractional delay control system

$$ {}^{C}D_t^{\alpha}u(t) + Au(t) = F(t, W_\delta(t)) + V_\sigma(t) \tag{1}$$

subject to the fractional nonlocal condition

$$ {}^{L}D_t^{1-\alpha}[u(0) - u_0] = h[u(t)], \tag{2}$$

where the unknown $u(\cdot)$ takes its values in a Hilbert space H with norm $\|\cdot\|$, ${}^{C}D_t^{\alpha}$ and ${}^{L}D_t^{1-\alpha}$ are the Caputo and Riemann-Liouville fractional derivatives with $0 < \alpha \leq 1$ and $t \in J = [0, a]$, respectively. Let $-A$ be a closed linear operator defined on a dense set S that generates a C_0-semigroup $Q(t),\ t \geq 0$, of bounded operators on H, $u_0 \in S$. We assume that $W_\delta = (A_1 u_{\delta_1}, \ldots, A_p u_{\delta_p})$ and $V_\sigma = (B_1 \mu_{\sigma_1} + \cdots + B_q \mu_{\sigma_q})$ are such that $\{A_i(t) : i = 1, \ldots, p,\ t \in J\}$ is a family of closed linear operators defined on dense sets $S_1, \ldots, S_p \supset S$ from H into H and $\{B_j(t) : U \to H,\ j = 1, \ldots, q,\ t \in J\}$ is a family of bounded linear operators. The control function μ belongs to the space $L^2(J, U)$, a Hilbert space of admissible control functions with U as a Hilbert space, $\delta_i, \sigma_j : J \to J'$ are delay arguments, $i = 1, \ldots, r$, $j = 1, \ldots, s$, and $J' = [0, t]$. The operators $F : J \times H^p \to H$ and $h : C(J : H) \to H$ are given abstract functions.

During the last decades, fractional differential equations have attract the attention of many mathematicians, physicists and engineers — see, e.g., Agarwal et al. (2010), Debbouche

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(2011a), Debbouche et al. (2012), El-Sayed (1996), Lakshmikantham and Vatsala (2008), Wang and Zhou (2011). The reason is that real phenomena, such as dielectric and electrode-electrolyte polarization, electromagnetic waves, earthquakes, fluid dynamics, traffic, viscoelasticity and viscoplasticity, can be described successfully and more accurately using fractional models — see Kilbas et al. (2006), Mozyrska and Torres (2010, 2011), Podlubny (1999), Samko et al. (1993). Fractional evolution equations with nonlocal conditions have been studied in many works — see Debbouche (2010, 2011b), N'Guérékata (2009), Zhou and Jiao (2010a) and references therein. Existence results to evolution equations with nonlocal conditions in a Banach space were first studied in Byszewski (1991a,b). In Deng (1993) it is shown that, using the nonlocal condition $u(0)+h(u)=u_0$ to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube, can give better results than using the standard local Cauchy condition $u(0)=u_0$. According to Deng (1993), function h takes the form $h(u)=\sum_{k=1}^{p} c_k u(t_k)$, where c_k, $k=1,\dots,p$, are given constants and $0 \le t_1 < \cdots < t_p \le a$. In Wang et al. (2012a) the controllability of semilinear fractional differential equations with nonlocal conditions in a Banach space is investigated through the Mönch fixed point theorem, where the semigroup generated by the linear part is not necessarily compact but the nonlinear term satisfies some weak compactness condition. The existence of mild solutions for impulsive fractional evolution Cauchy problems involving Caputo fractional derivatives is discussed in Wang et al. (2011) by means of the theory of operators semigroup and probability density functions via impulsive conditions, while the solvability and optimal control of a class of fractional integrodifferential evolution systems with an infinite delay in Banach spaces is studied in Wang et al. (2012c): a concept for solution is introduced, existence and continuous dependence of solutions are investigated, and existence of optimal controls proved. In Wang et al. (2012d) optimal feedback control laws for Lagrange problems subject to semilinear fractional-order systems in Banach spaces are established.

Exact controllability of fractional order systems has been proved by many authors: Arjunan and Kavitha (2011), Debbouche and Baleanu (2011, 2012), Triggiani (1977), Wang et al. (2012b). The main tool is to convert the controllability question into a fixed point problem with the assumption that the controllability operator has an induced inverse on a quotient space. To prove controllability, an assumption that the semigroup (resp. resolvent operator) associated with the linear part is compact is then often made. However, if the compactness condition holds on the bounded operator that maps the control function or the generated C_0-semigroup, then the controllability operator is also compact and its inverse does not exist if the state space is infinite dimensional — see Triggiani (1977). Thus, the concept of exact controllability is too strong in infinite dimensional spaces and the approximate controllability notion is more appropriate.

Approximate controllability of integer order systems has been proved in several works. In contrast, papers dealing with the approximate controllability of fractional order systems are scarce. Recently, the subject was addressed in Sakthivel et al. (2011), Sakthivel and Ren (2012), while sufficient conditions for the (delay) approximate controllability of fractional order systems, in which the nonlinear term depends on both state and control variables, are investigated in Kumar and Sukavanam (2012), Sukavanam and Kumar (2011), and the case of partial neutral fractional functional differential systems with a state-dependent delay is considered in Yan (2012).

Our main objective is to study the approximate controllability of semilinear fractional control systems, where the control function depends on multi-delay arguments and where the nonlocal condition is fractional. The result is obtained under the assumption that the associated linear system is approximately controllable. In particular, the controllability question is transformed to a fixed point problem for an appropriate nonlinear operator in a function space. For that we need to construct a suitable set of sufficient conditions. The paper is organized as follows: in Section 2, we present some essential definitions of fractional calculus and basic facts in the semigroup theory that will be used to obtain our main results. In Section 3, we state and prove existence and approximate controllability results for problem (1)–(2). Finally, in Section 4 we

illustrate the new results of the paper with an example.

2 Preliminaries

In this section, we introduce some basic definitions, notations and lemmas, which will be used throughout the work. In particular, we give necessary properties of fractional calculus (see Kilbas et al. (2006), Podlubny (1999), Samko et al. (1993)) and some fundamental facts in semigroup theory (see Hille and Phillips (1957), Pazy (1983), Zaidman (1979)).

Let $(H, \|\cdot\|)$ be a Hilbert space, $C(J,H)$ denote the Hilbert space of continuous functions from J into H with the norm $\|u\|_J = \sup\{\|u(t)\| : t \in J\}$, and $L(H)$ be the Hilbert space of bounded linear operators from H to H. Further, let $E(H)$ be the space of all bounded linear operators from H to H with the norm $\|G\|_{E(H)} = \sup\{\|G(u)\| : \|u\| = 1\}$, where $G \in E(H)$ and $u \in H$. Throughout the paper, let $-A$ be the infinitesimal generator of the C_0-semigroup $Q(t)$, $t \geq 0$, of uniformly bounded linear operators on H. Clearly, $M = \sup_{t\in[0,\infty)} \|Q(t)\| < \infty$.

Definition 2.1: The fractional integral of order $\alpha > 0$ of a function $f \in C([0,\infty))$ is given by

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,$$

where Γ is the gamma function, provided the right-hand side is point-wise defined on $[0,\infty)$.

Definition 2.2: The Riemann–Liouville derivative of order $\alpha > 0$ of a function $f \in C([0,\infty))$ is given by

$${}^L D^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,$$

where $n \in \mathbb{N}$ is such that $n-1 < \alpha < n$.

Definition 2.3: The Caputo derivative of order $\alpha > 0$ of a function $f \in C([0,\infty))$ is given by

$${}^C D^\alpha f(t) := {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0,$$

where $n \in \mathbb{N}$ is such that $n-1 < \alpha < n$.

Remark 1: The following properties hold:

(1) If $f \in C^n([0,\infty))$, then

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^n(t), \quad t > 0, \quad n-1 < \alpha < n.$$

(2) The Caputo derivative of a constant is equal to zero.

(3) If f is an abstract function with values in H, then the integrals which appear in Definitions 2.1–2.3 are taken in Bochner's sense.

According to previous definitions, it is suitable to rewrite the problem (1)–(2) in the equivalent integral form

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [-Au(s) + F(s, W_\delta(s)) + V_\sigma(s)] ds. \tag{3}$$

Remark 2: We note that:

(1) For the nonlocal condition, the function $u(0)$ is dependent on t.
(2) The Riemann–Liouville fractional derivative of $u(0)-u_0$ is well defined and ${}^L D_t^{1-\alpha} u_0 \neq 0$.
(3) The function $u(0)$ takes the form $u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{h[u(s)]}{(t-s)^\alpha} ds$, where $u(0) - u_0|_{t=0} = v_0$.
(4) The explicit and implicit integrals given in (3) exist (taken in Bochner's sense).

Definition 2.4: A state $u \in C(J, H)$ is a mild solution of (1)–(2) if, for each control $\mu \in L^2(J, U)$, it satisfies the following integral equation:

$$u(t) = S_\alpha(t) \left[u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{h[u(s)]}{(t-s)^\alpha} ds \right] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[F(s, W_\delta(s)) + V_\sigma(s)] ds,$$

where

$$S_\alpha(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) d\theta, \quad T_\alpha(t) = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) Q(t^\alpha \theta) d\theta,$$

$$\zeta_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0, \quad \varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha), \theta \in (0, \infty).$$

Remark 3: In Definition 2.4, ζ_α is a probability density function defined on $(0, \infty)$, that is, $\zeta_\alpha(\theta) \geq 0$, $\theta \in (0, \infty)$, and $\int_0^\infty \zeta_\alpha(\theta) d\theta = 1$ (compare with Debbouche and El-Borai (2009), El-Borai (2002); see also Zhou and Jiao (2010a,b)).

The following lemma can be found in Debbouche and El-Borai (2009), Zhou and Jiao (2010b).

Lemma 2.5: *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:*

(1) For any fixed $t \geq 0$, the operators $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded, i.e., for any $u \in H$, $\|S_\alpha(t)u\| \leq M\|u\|$ and $\|T_\alpha(t)u\| \leq \frac{M\alpha}{\Gamma(1+\alpha)}\|u\|$.
(2) $\{S_\alpha(t), t \geq 0\}$ and $\{T_\alpha(t), t \geq 0\}$ are strongly continuous, i.e., for $u \in H$ and $0 \leq t_1 < t_2 \leq a$, one has $\|S_\alpha(t_2)u - S_\alpha(t_1)u\| \to 0$ and $\|T_\alpha(t_2)u - T_\alpha(t_1)u\| \to 0$ as $t_1 \to t_2$.

Motivated by the recent works of Debbouche and Baleanu (2011), Kumar and Sukavanam (2012), Sakthivel and Ren (2012), Sakthivel et al. (2011), Yan (2012), we make use of the following notions and lemmas.

Let $u_a(u(0); \mu)$ be the state value of (1)–(2) at terminal time a, corresponding to the control μ and the nonlocal value $u(0)$. For every $u_0, v_0 \in H$, we introduce the set

$$\mathfrak{R}(a, u(0)) := \left\{ u_a \left(u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{h[u(s)]}{(t-s)^\alpha} ds; \mu \right)(0) : \mu(\cdot) \in L^2(J, U) \right\},$$

which is called the reachable set associated with (1)–(2) at terminal time a. Its closure in H is denoted by $\overline{\mathfrak{R}(a, u(0))}$.

Definition 2.6: The system (1)–(2) is said to be approximately controllable on J if $\overline{\mathfrak{R}(a, u(0))} = H$, that is, given an arbitrary $\epsilon > 0$, it is possible to steer in time a the system from point $u(0)$ to all points in the state space H within a distance ϵ.

Consider the following linear nonlocal fractional multi-delay control system:

$$ {}^C D_t^\alpha u(t) + A u(t) = B_1 \mu(\sigma_1(t)) + \cdots + B_q \mu(\sigma_q(t)), \tag{4}$$

$$^{L}D_t^{1-\alpha}[u(0)-u_0]=h[u(t)]. \tag{5}$$

The approximate controllability for the linear fractional nonlocal multi-delay control system (4)–(5) is a natural generalization of the notion of approximate controllability of a linear first-order control system ($\alpha=1$, $\sigma_j(t)=t$, $j=1$ and $h=0$). It is convenient at this point to introduce the multi-delay controllability operator associated with (4)–(5). One has

$$\Gamma^{a}_{0,\sigma_j}=\int_0^a (a-s)^{\alpha-1}T_\alpha(a-s)B_jB_j^{*}T_\alpha^{*}(a-s)ds, \quad j=1,\dots,q,$$

where B_j^* denotes the adjoint of B_j and $T_\alpha^*(t)$ is the adjoint of $T_\alpha(t)$. We introduce here the new operator $\Gamma^a_{0,\sigma}=\Gamma^a_{0,\sigma_1,\dots,\sigma_q}$ of multi-delay controllability associated with (4)–(5) as

$$\Gamma^a_{0,\sigma}:=\int_0^a (a-s)^{\alpha-1}T_\alpha(a-s)[B_1B_1^*+\cdots+B_{q-1}B_{q-1}^*+B_qB_q^*]T_\alpha^*(a-s)ds.$$

It is straightforward to see that $\Gamma^a_{0,\sigma}$ is a linear bounded positive operator. The following lemma is proved in Bashirov and Mahmudov (1999) for linear positive operators in Hilbert spaces, while a Banach space version is given in Mahmudov (2003). See also Sakthivel et al. (2011), Sakthivel and Ren (2012).

Lemma 2.7: *Let $\mathcal{R}(\beta,\Gamma^a_{0,\sigma})=(\beta I+\Gamma^a_{0,\sigma})^{-1}$ for $\beta>0$. The linear fractional control system (4)–(5) is approximately controllable on J if and only if $\beta\mathcal{R}(\beta,\Gamma^a_{0,\sigma})\to 0$ as $\beta\to 0^+$ in the strong operator topology.*

In the sequel we use Schauder's fixed point theorem that can be found in any standard textbook of Functional Analysis (see, e.g., (Granas and Dugundji 2003, Theorem 3.2, p. 119) or (Smart 1974, Theorem 4.1.1, p. 25)):

Lemma 2.8: *If Ω is a closed bounded and convex subset of a Banach space X and $\psi:\Omega\to\Omega$ is completely continuous, then ψ has a fixed point in Ω.*

3 Main results

We now formulate and establish our results on the approximate controllability of the nonlocal delay system (1)–(2). With this purpose, firstly we prove the existence of solutions for the fractional control system (1)–(2) by using Schauder's fixed point theorem (Lemma 2.8). Then, we show that under certain assumptions, the approximate controllability of the fractional system (1)–(2) is implied by the approximate controllability of the corresponding linear system (4)–(5). Before starting, we make the following hypotheses:

- (H_1) The semigroup $Q(t)$ is a compact operator for $t>0$.
- (H_2) For each $t\in J$, the function $F(t,\cdot):S_1\times\cdots\times S_p\to H$ is continuous and for each $W_\delta\in C[(J',S_1)\times\cdots\times(J',S_p);H]$, in particular, for every element $u\in\cap_i S_i$, $i=1,\dots,p$, the function $F(\cdot,W_\delta):J\to H$ is strongly measurable.
- (H_3) There exist functions $m_i\in L^{\frac{1}{1-\alpha}}(J',\mathbb{R}^+)$ such that $|F(t,W_\delta(t))|\le m_1(\delta_1(t))+\cdots+m_p(\delta_p(t))$ for all $u\in\cap_i S_i$, $0<\alpha<1$, $i=1,\dots,p$, and almost all $t\in J'$.
- (H_4) The function $h:C(J:H)\to H$ is bounded in H, that is, there exists a constant $k_1>0$ such that $\|h(u)\|_H\le k_1$.
- (H_5) The delay arguments $\delta_i,\sigma_j:J\to J'$ are absolutely continuous and satisfy $|\delta_i(t)|\le t$ and $|\sigma_j(t)|\le t$, for every $t\in J$, $i=1,\dots,r$ and $j=1,\dots,s$.
- (H_6) The function $F:J\times H^p\to H$ is continuous and uniformly bounded and there exist $N_{\delta_1},\dots,N_{\delta_p}>0$ such that $\|F(t,W_\delta(t))\|\le N_{\delta_1}+\cdots+N_{\delta_p}$ for all $(t,W_\delta)\in J\times H^p$.

For the proof of our Theorem 3.2, we make use of Lemma 3.1 whose proof can be found in Zhou and Jiao (2010b).

Lemma 3.1: *If the assumption* (H_1) *is satisfied, then* $S_\alpha(t)$ *and* $T_\alpha(t)$ *are also compact operators for every* $t > 0$.

Theorem 3.2: *If the hypotheses* (H_1)*–*(H_5) *are satisfied, then the fractional nonlocal semilinear delay control system* (1)–(2) *has a mild solution on* J*; here* $M_i = \|m_i\|_{L^{\frac{1}{1-\alpha}}(J')}$, $i = 1, \ldots, p$, *and* $M_{B_j} = \|B_j\|$, $j = 1, \ldots, q$.

Proof Consider the set

$$S_r := \left\{ u \in C(J,H) | u(0) = u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t h[u(s)](t-s)^{-\alpha} ds, \|u\| \leq r \right\},$$

where r is a positive constant. For $\beta > 0$, we define the operator ψ_β on $C(J,X)$ as follows: $(\psi_\beta u)(t) := z(t)$ with

$$z(t) := S_\alpha(t)u(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[F(s, W_\delta(s)) + B_1 v(\sigma_1(s)) + \cdots + B_q v(\sigma_q(s))] ds,$$

$v(\sigma_1(\cdot)) := B_1^*, \ldots, v(\sigma_{q-1}(\cdot)) := B_{q-1}^*$, $v(\sigma_q(t)) := B_q^* T_\alpha^*(a-t)\mathcal{R}(\beta, \Gamma_{0,\sigma}^a) p(u(\cdot))$, and

$$p(u(\cdot)) := u_a - S_\alpha(a)u(0) - \int_0^a (a-s)^{\alpha-1} T_\alpha(a-s) F(s, W_\delta(s)) ds.$$

In order to show that for all $\beta > 0$ the operator ψ_β from $C(J,H)$ into itself has a fixed point, we divide the proof into several steps. *Step 1.* For $\beta > 0$, there is a positive constant $r_0 = r(\beta)$ such that $\psi_\beta : S_{r_0} \to S_{r_0}$. For any positive constant r and $u \in S_r$, and since $W_\delta(t)$ is continuous in t, according to assumption (H_2) $F(t, W_\delta(t))$ is a measurable function on J as well as function $(t-s)^{\alpha-1} \in L^{\frac{1}{\alpha}}(J')$. Using (1) of Lemma 2.5, (H_3) and Hölder's inequality, we get:

$$\begin{aligned} \|z(t)\| &\leq M\|u(0)\| + \frac{M\alpha}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \|F(s, W_\delta(s)) + B_1 v(\sigma_1(s)) + \cdots + B_q v(\sigma_q(s))\| ds \\ &\leq M \left[\|u_0\| + \|v_0\| + \frac{k_1 a^{1-\alpha}}{\Gamma(2-\alpha)} \right] \\ &\quad + \frac{\alpha M \alpha^\alpha a^{2\alpha-1}}{\Gamma(1+\alpha)(2\alpha-1)^\alpha} \left[M_1 + \cdots + M_p + M_{B_1}^2 + \cdots + M_{B_{q-1}}^2 + M_{B_q} \|v(\sigma_q(s))\| \right] \end{aligned}$$

and

$$\|v(\sigma_q(t))\| = \frac{1}{\beta} M_{B_q} M \left[\|u_a\| + M\|u(0)\| + \frac{\alpha M \alpha^\alpha a^{2\alpha-1} (M_1 + \cdots + M_p)}{\Gamma(1+\alpha)(2\alpha-1)^\alpha} \right].$$

We deduce that for large enough $r_0 > 0$, the inequality $\|(\psi_\beta u)(t)\| \leq r_0$ holds, i.e., $(\psi_\beta u) \in S_{r_0}$. Therefore, ψ_β maps S_{r_0} into itself. *Step 2.* For each $0 < \alpha \leq 1$, the operator ψ_β maps S_{r_0} into a relatively compact subset of S_{r_0}. Based on the infinite-dimensional version of the Ascoli-Arzela theorem, we show that: (i) the set $V(t) := \{(\psi_\beta u)(t) : u(\cdot) \in S_{r_0}\}$ is relatively compact in H for any $t \in J$; (ii) the family of functions $\{(\psi_\beta u), u \in S_{r_0}\}$ is relatively compact (for this, it suffices to prove that $V(t)$ is bounded and equicontinuous). We begin by proving (i). Let t be a fixed

real number, and let τ be a given real number satisfying $0 \le \tau < t$. For any $\eta > 0$, define

$$\begin{aligned}
(\psi_\beta^{\tau,\eta}u)(t) = &\int_\eta^\infty \zeta_\alpha(\theta)Q(t^\alpha\theta)\left[u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^{t-\tau}(t-s)^{-\alpha}h(u(s))ds\right]d\theta \\
&+ \alpha\int_0^{t-\tau}\int_\eta^\infty (t-s)^{\alpha-1}\theta\zeta_\alpha(\theta)Q\left((t-s)^\alpha\theta\right)F(s,W_\delta(s))d\theta ds \\
&+ \alpha\int_0^{t-\tau}\int_\eta^\infty (t-s)^{\alpha-1}\theta\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)V_\sigma(s)d\theta ds \\
= &Q(\tau^\alpha\eta)\int_\eta^\infty \zeta_\alpha(\theta)Q(t^\alpha\theta-\tau^\alpha\eta)\left[u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^{t-\tau}(t-s)^{-\alpha}h(u(s))ds\right]d\theta \\
&+ Q(\tau^\alpha\eta)\alpha\int_0^{t-\tau}\int_\eta^\infty (t-s)^{\alpha-1}\theta\zeta_\alpha(\theta)Q((t-s)^\alpha\theta-\tau^\alpha\eta)F(s,W_\delta(s))d\theta ds \\
&+ Q(\tau^\alpha\eta)\alpha\int_0^{t-\tau}\int_\eta^\infty (t-s)^{\alpha-1}\theta\zeta_\alpha(\theta)Q((t-s)^\alpha\theta-\tau^\alpha\eta)V_\sigma(s)d\theta ds \\
:= &Q(\tau^\alpha\eta)y(t,\tau).
\end{aligned}$$

Because $Q(\tau^\alpha\eta)$ is compact and $y(t,\tau)$ is bounded on S_{r_0}, $\{(\psi_\beta^{\tau,\eta}u)(t) : u(\cdot) \in S_{r_0}\}$ is a relatively compact set in H. On the other hand,

$$\begin{aligned}
&\|(\psi_\beta u)(t) - (\psi_\beta^{\tau,\eta}u)(t)\| \\
&= \Bigg\|\int_0^\eta \zeta_\alpha(\theta)Q(t^\alpha\theta)\left[u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}h(u(s))ds\right]d\theta \\
&\quad + \int_\eta^\infty \zeta_\alpha(\theta)Q(t^\alpha\theta)\left[u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)}\int_{t-\tau}^t (t-s)^{-\alpha}h(u(s))ds\right]d\theta\Bigg\| \\
&\quad + \alpha\Bigg\|\int_0^t\int_0^\eta (t-s)^{\alpha-1}\theta\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)[F(s,W_\delta(s)) + V_\sigma(s)]d\theta ds \\
&\quad + \int_{t-\tau}^t\int_\eta^\infty (t-s)^{\alpha-1}\theta\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)[F(s,W_\delta(s)) + V_\sigma(s)]d\theta ds\Bigg\| \\
&\le M\left\{\left[\|u_0\| + \|v_0\| + \frac{k_1 a^{1-\alpha}}{\Gamma(2-\alpha)}\right]\int_0^\eta \zeta_\alpha(\theta)d\theta + \left[\|u_0\| + \|v_0\| + \frac{k_1\tau^{1-\alpha}}{\Gamma(2-\alpha)}\right]\right\} \\
&\quad + \frac{\alpha M\alpha^\alpha a^{2\alpha-1}}{\Gamma(1+\alpha)(2\alpha-1)^\alpha}\Bigg\{M_1 + \cdots + M_p + M_{B_1}^2 + \cdots + M_{B_{q-1}}^2 \\
&\quad + \frac{1}{\beta}M_{B_q}^2 M\left[\|u_a\| + M\|u(0)\| + \frac{\alpha M\alpha^\alpha a^{2\alpha-1}(M_1 + \cdots + M_p)}{\Gamma(1+\alpha)(2\alpha-1)^\alpha}\right]\Bigg\}\int_0^\eta \theta\zeta_\alpha(\theta)d\theta \\
&\quad + \frac{\alpha M\alpha^\alpha \tau^{2\alpha-1}}{\Gamma(1+\alpha)(2\alpha-1)^\alpha}\Bigg\{M_1 + \cdots + M_p + M_{B_1}^2 + \cdots + M_{B_{q-1}}^2 \\
&\quad + \frac{1}{\beta}M_{B_q}^2 M\left[\|u_a\| + M\|u(0)\| + \frac{\alpha M\alpha^\alpha a^{2\alpha-1}(M_1 + \cdots + M_p)}{\Gamma(1+\alpha)(2\alpha-1)^\alpha}\right]\Bigg\}.
\end{aligned}$$

This implies that there are relatively compact sets arbitrarily close to $V(t)$ for each $t \in (0,a]$. Hence, $V(t)$, $t \in (0,a]$, is relatively compact in H. We now prove (ii). First we show that $V(t) := \{(\psi_\beta u)(\cdot) : u(\cdot) \in S_{r_0}\}$ is an equicontinuous family of functions on $[0,a]$. For any $u \in S_{r_0}$

and $0 \leq t_1 \leq t_2 \leq a$,

$$
\begin{aligned}
\|z(t_2) - z(t_1)\| &\leq \left\| S_\alpha(t_2)[u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} h(u(s)) ds] \right\| \\
&+ \left\| S_\alpha(t_2)[u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} [(t_2 - s)^{-\alpha} - (t_1 - s)^{-\alpha}] h(u(s)) ds] \right\| \\
&+ \left\| [S_\alpha(t_2) - S_\alpha(t_1)][u_0 + v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} (t_1 - s)^{-\alpha} h(u(s)) ds] \right\| \\
&+ \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) \left[F(s, W_\delta(s)) + B_1 v(\sigma_1(s)) + \cdots + B_q v(\sigma_q(s)) \right] ds \right\| \\
&+ \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] T_\alpha(t_2 - s) \left[F(s, W_\delta(s)) + B_1 v(\sigma_1(s)) + \cdots + B_q v(\sigma_q(s)) \right] ds \right\| \\
&+ \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [T_\alpha(t_2 - s) - T_\alpha(t_1 - s)] \left[F(s, W_\delta(s)) + B_1 v(\sigma_1(s)) + \cdots + B_q v(\sigma_q(s)) \right] ds \right\| \\
&\leq I_1 + I_2 + I_3 + I_1^* + I_2^* + I_3^*.
\end{aligned}
$$

We have

$$
\begin{gathered}
I_1 \leq M \left\{ \|u_0\| + \|v_0\| + \frac{k_1 (t_2 - t_1)^{1-\alpha}}{\Gamma(2-\alpha)} \right\}, \\
I_2 \leq M \left\{ \|u_0\| + \|v_0\| + \frac{k_1 [(t_2 - t_1)^{1-\alpha} + t_2^{1-\alpha} + t_1^{1-\alpha}]}{\Gamma(2-\alpha)} \right\}, \\
I_3 \leq 2M \left\{ \|u_0\| + \|v_0\| + \frac{k_1 t_1^{1-\alpha}}{\Gamma(2-\alpha)} \right\}.
\end{gathered}
$$

Using Hölder's inequality and assumption (H_3), one gets

$$
\begin{aligned}
I_1^* &\leq \frac{\alpha M \alpha^\alpha (t_2 - t_1)^{2\alpha-1}}{\Gamma(1+\alpha)(2\alpha-1)^\alpha} \left[M_1 + \cdots + M_p + M_{B_1}^2 + \cdots + M_{B_{q-1}}^2 + M_{B_q} \|v(\sigma_q)\| \right], \\
I_2^* &\leq \frac{\alpha M \alpha^\alpha (t_2 - t_1)^{2\alpha-1}}{\Gamma(1+\alpha)(2\alpha-1)^\alpha} \left[M_1 + \cdots + M_p + M_{B_1}^2 + \cdots + M_{B_{q-1}}^2 + M_{B_q} \|v(\sigma_q)\| \right].
\end{aligned}
$$

Obviously, $I_3^* = 0$ for $t_1 = 0$ and $0 < t_2 \leq a$. For $t_1 > 0$ and $\epsilon > 0$ small enough, we obtain

$$
\begin{aligned}
I_3^* \leq & \int_0^{t_1-\epsilon} (t_1 - s)^{\alpha-1} \|T_\alpha(t_2 - s) - T_\alpha(t_1 - s)\| \\
& \times [\|F(s, W_\delta(s))\| + \|B_1 v(\sigma_1(s))\| + \cdots + \|B_q v(\sigma_q(s))\|] ds \\
& + \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{\alpha-1} \|T_\alpha(t_2 - s) - T_\alpha(t_1 - s)\| \\
& \times [\|F(s, W_\delta(s))\| + \|B_1 v(\sigma_1(s))\| + \cdots + \|B_q v(\sigma_q(s))\|] ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\alpha^\alpha[t_1^{\frac{2\alpha-1}{\alpha}} - \epsilon^{\frac{2\alpha-1}{\alpha}}]^\alpha}{\Gamma(1+\alpha)(2\alpha-1)^\alpha}\left[M_1 + \cdots + M_p + M_{B_1}^2 + \cdots + M_{B_{q-1}}^2 + M_{B_q}\|v(\sigma_q)\|\right] \\
&\quad \times \sup_{s\in[0,t_1-\epsilon]} \|T_\alpha(t_2-s) - T_\alpha(t_1-s)\| \\
&\quad + \frac{2\alpha M\alpha^\alpha\epsilon^{2\alpha-1}}{\Gamma(1+\alpha)(2\alpha-1)^\alpha}\left[M_1 + \cdots + M_p + M_{B_1}^2 + \cdots + M_{B_{q-1}}^2 + M_{B_q}\|v(\sigma_q)\|\right].
\end{aligned}
$$

Note that $I_1, I_2, I_3, I_1^*, I_2^* \to 0$ as $t_2 - t_1 \to 0$. Moreover, the assumption (H_1) together with Lemma 3.1 imply the continuity of $T_\alpha(t)$ in t in the uniform operator topology. It is easy to verify that I_3^* tends to zero independently of $u \in S_{r_0}$ as $t_2 - t_1 \to 0, \epsilon \to 0$. Consequently, $I_1 + I_2 + I_3 + I_1^* + I_2^* + I_3^*$ does not depend on the particular choices of $u(\cdot)$ and tends to zero as $t_2 - t_1 \to 0$, which means that $\{(\psi_\beta u), u \in S_{r_0}\}$ is equicontinuous. Therefore, $\psi_\beta[S_{r_0}]$ is equicontinuous and also bounded. By the Ascoli–Arzela theorem, $\psi_\beta[S_{r_0}]$ is relatively compact in $C(J,H)$. On the other hand, it is easy to see that for all $\beta > 0$, ψ_β is continuous on $C(J,H)$. Hence, for all $\beta > 0$, ψ_β is a completely continuous operator on $C(J,H)$. According to Schauder's fixed point theorem (Lemma 2.8), ψ_β has a fixed point. Thus, the fractional nonlocal control system (1)–(2) has a mild solution on J. □

Theorem 3.3: *Assume that (H_1)–(H_6) are satisfied and the linear system* (4)–(5) *is approximately controllable on J. Then the semilinear fractional nonlocal delay system* (1)–(2) *is approximately controllable on J.*

Proof Let $\hat{u}_\beta(\cdot)$ be a fixed point of ψ_β in S_{r_0}. By Theorem 3.2, any fixed point of ψ_β is a mild solution of (1)–(2) under the multi-delay controls

$$
\begin{cases}
\hat{\mu}_\beta(\sigma_1(\cdot)) = B_1^*, \\
\qquad \vdots \\
\hat{\mu}_\beta(\sigma_{q-1}(\cdot)) = B_{q-1}^*, \\
\hat{\mu}_\beta(\sigma_q(t)) = B_q^* T_\alpha^*(a-t)\mathcal{R}(\beta, \Gamma_{0,\sigma}^a)p(\hat{u}_\beta),
\end{cases}
$$

and satisfies

$$
\hat{u}_\beta(a) = u_a + \beta\mathcal{R}(\beta, \Gamma_{0,\sigma}^a)p(\hat{u}_\beta). \tag{6}
$$

From condition (H_6), it follows that

$$
\int_0^a \|F(s, \hat{W}_\beta(\delta(s))\|^2 ds \leq a[N_{\delta_1} + \cdots + N_{\delta_p}]^2,
$$

where $\hat{W}_\beta(\delta) = (A_1\hat{u}_\beta(\delta_1), \ldots, A_p\hat{u}_\beta(\delta_p))$. Consequently, the sequence $\{F(t, \hat{W}_\beta(\delta(t))\}_{t\in J}$ is bounded in $L_2(J,H)$. Then there is a subsequence, denoted by $\{F_k(t, \hat{W}_\beta(\delta(t))\}_{t\in J}$, that converges weakly to a function $f(t)$ in $L_2(J,H)$. Define

$$
\omega := u_a - S_\alpha(a)u(0) - \int_0^a (a-s)^{\alpha-1}T_\alpha(a-s)f(s)ds.
$$

It follows that

$$
\begin{aligned}
\|p(\hat{u}_\beta) - \omega\| &= \left\|\int_0^a (a-s)^{\alpha-1}T_\alpha(a-s)[F_k(t, \hat{W}_\beta(\delta(t)) - f(s)]ds\right\| \\
&\leq \sup_{t\in J}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(a-s)[F_k(t, \hat{W}_\beta(\delta(t)) - f(s)]ds\right\|.
\end{aligned} \tag{7}
$$

Similarly as in the proof of Theorem 3.2, using the infinite-dimensional version of the Ascoli–Arzela theorem one can show that the operator $l(\cdot) \to \int_0^{\cdot} (\cdot - s) T_\alpha(\cdot - s) l(s) ds : L_2(J, H) \to C(J, H)$ is compact. Consequently, the right-hand side of the inequality (7) tends to zero as $\beta \to 0^+$. Then, from (6), we obtain that

$$\begin{aligned} \|\hat{u}_\beta(a) - u_a\| &= \|\beta \mathcal{R}(\beta, \Gamma_{0,\sigma}^a)(\omega)\| + \|\beta \mathcal{R}(\beta, \Gamma_{0,\sigma}^a)\| \|p(\hat{u}_\beta) - \omega\| \\ &\leq \|\beta \mathcal{R}(\beta, \Gamma_{0,\sigma}^a)(\omega)\| + \|p(\hat{u}_\beta) - \omega\| \to 0. \end{aligned} \tag{8}$$

Using the inequality (7) and Lemma 2.7, we conclude that (8) tends to zero as $\beta \to 0^+$. Hence, system (1)–(2) is approximately controllable on J. □

4 An example

Consider the fractional nonlocal partial multi-delay control system

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a(x) \frac{\partial^2 u(x,t)}{\partial x^2} = \Phi(t, D_x^p u(x, \delta_p(t))) + B^q \mu(x, \sigma_q(t)) \tag{9}$$

subject to

$$u(x, 0) = g(x) + \sum_{k=1}^{m} \frac{c_k}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{u(x, s_k)}{(t_k - s_k)^\alpha} ds_k, \quad x \in [0, \pi], \tag{10}$$

$$u(0, t) = u(\pi, t) = 0, \ t \in J, \tag{11}$$

where $0 < \alpha \leq 1$, $0 < t_1 < \cdots < t_m < a$, Φ is given as F, and the function $a(x)$ is continuous. Let us take the operators D_x^p and B^q as follows:

$$\begin{aligned} D_x^p u(x, \delta_p(t)) &= (\partial_x u(x, \sin t), \partial_x^2 u(x, \sin t/2), \ldots, \partial_x^p u(x, \sin t/p)), \\ B^q \mu(x, \sigma_q(t)) &= \xi(x, \sin t) + \xi(x, \sin t/2) + \cdots + \xi(x, \sin t/q), \end{aligned}$$

such that the control function is $\mu(t) = \xi(\cdot, t)$, where $\xi : [0, \pi] \times J \to [0, \pi]$ is continuous, the multi-delays $\delta_\tau(t) = \sigma_\tau(t) = \sin(t/\tau)$, $\tau = 1, \ldots, \eta$, $\eta = \max(p, q)$, and the nonlocal function is given by $h(u(\cdot, t)) = \sum_{k=1}^m c_k u(\cdot, t_k)$. Assume that

$$H = L^2[0, \pi], \ S_r = \{y \in L^2[0, \pi] : \|y\| \leq r\}.$$

We define $A : H \to H$ by $(Aw)(x) = a(x) w''$ with the domain

$$D(A) = \{w \in H : w, w' \text{ are absolutely continuous}, w'' \in H, w(0) = w(\pi) = 0\},$$

dense in the Hilbert space H. Then,

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A),$$

where $(\cdot, \cdot)$ is the inner product in $L^2[0, \pi]$ and $w_n(t) = \sqrt{\frac{2}{\pi}} \sin n t^\alpha$, $0 < \alpha \leq 1$, $t \in J$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvectors in A. It is well known that A generates a compact, analytic

and self-adjoint semigroup $\{Q(t),\, t \geq 0\}$ in H. For all $t \geq 0$ and $w \in H$,

$$Q(t)w = \sum_{n=1}^{\infty} e^{-n^2 t^\alpha} (w, w_n) w_n, \quad \|Q(t)\| \leq e^{-t}. \tag{12}$$

Therefore, the problem (9)–(11) is a formulation of the control system (1)–(2). Moreover, all the assumptions (H_1)–(H_6) hold. Then, the associated linear system of (9)–(11) is not exactly controllable but it is approximately controllable. Indeed, since the semigroup $Q(t)$ given by (12) is compact, then the controllability operator is also compact and hence the induced inverse does not exist because the state space is infinite dimensional. Thus, the concept of exact controllability is too strong. On the other hand, we have $\lambda \mathcal{R}(\lambda, \Gamma_{0,i}^{a}) \to 0$ as $\lambda \to 0^+$, $i = 1, 2$, in the strong operator topology, which is a necessary and sufficient condition for the linear system to be approximately controllable. Hence, by Theorems 3.2 and 3.3, the control system (9)–(11) is approximately controllable on J.

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